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A bracket representation of the monoid of links

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1 Introduction

A *knot* is an oriented simple closed curve embedded in the 3-dimensional Euclidian space $\mathbf{E}^3$. A mutually disjoint union of knots is called a *link*. We say that two links are *equivalent* if one can be transformed to the other by some homeomorphism in $\mathbf{E}^3$. If two links L_1 and L_2 are equivalent, then we write $L_1 \sim L_2$. By $\mathcal{L}$, we denote the set of equivalence classes of links. We usually identify a link with its equivalence class. So an element of $\mathcal{L}$ is actually an equivalence class of links, but it is simply called a link. To handle links, we usually project them on the 2-dimensional plane (see Fig.1). A link projected on the 2-dimensional plane is called a *link diagram* of it. The link diagram of a link L is denoted by $D(L)$. Remark that there are infinitely many link diagrams of a link.

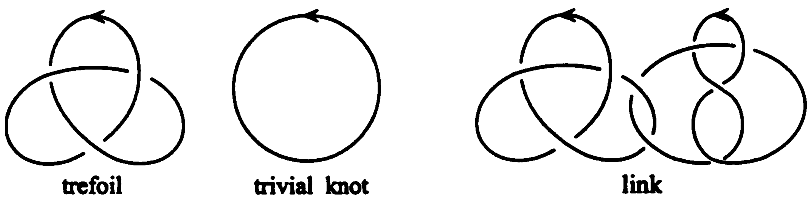
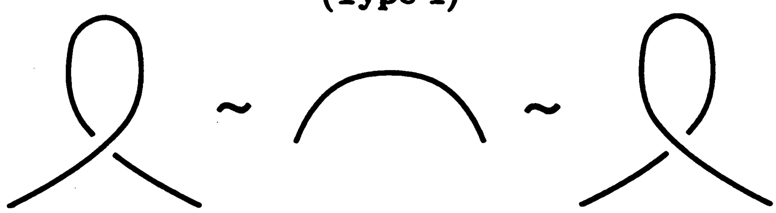


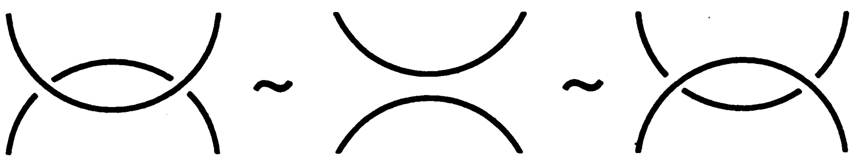
Fig.1

The following deformations of a part of a link diagram are called the *Reidemeister moves*.

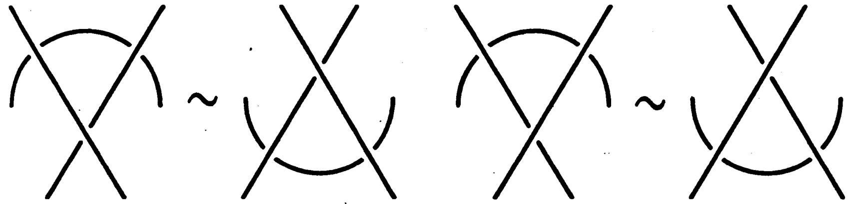
(Type 1)



(Type 2)



(Type 3)



In the link (or knot) theory, the following result is fundamental.

Result 1.1 *Two links are equivalent if and only if there is a finite sequence of the Reidemeister moves such that a link diagram of one of them can be deformed to a link diagram of the other one.*

In this paper, we assume that for each link one fixed component is chosen and a fixed point on this component is given. For each link L, a fixed point on a chosen component of L is called a *marked point* of L. A product of two links L_1 and L_2, denoted by $L_1 \# L_2$, is defined as follows. Let a_1 and a_2 be marked points of L_1 and L_2, respectively. If the orientations of $D(L_1)$ at a_1 and $D(L_2)$ at a_2 are opposite, then, remove a small arc from $D(L_1)$ containing a_1 and a small arc from $D(L_2)$ containing a_2. Let α and α' (resp. β and β') be two end points of $D(L_1)$ (resp. $D(L_2)$) removed the small arc. Then attach $D(L_1)$ and $D(L_2)$ at these points identifying α with β and α' with β'.

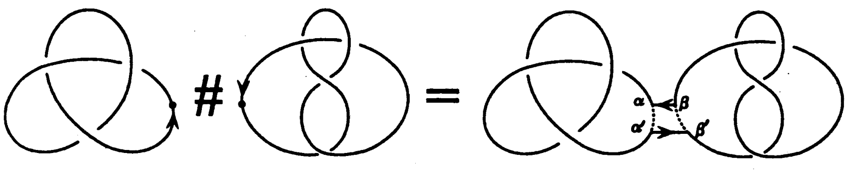


Fig.2

If the orientations of $D(L_1)$ at a_1 and $D(L_2)$ at a_2 agree, apply the deformation (Type 1) to $D(L_1)$ at a_1 or $D(L_2)$ at a_2, then the orientations of $D(L_1)$ at a_1 and $D(L_2)$ at a_2 become opposite. And then attach $D(L_1)$ and $D(L_2)$ as in the above way (see Fig.3).

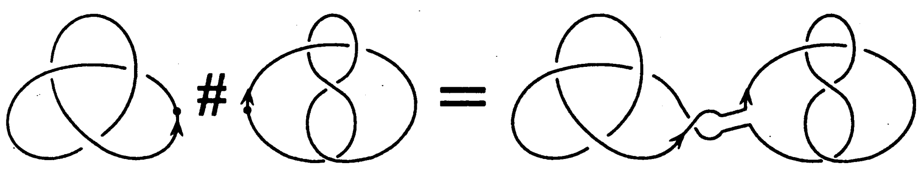


Fig.3

Remark that the above product does not depend on the choice of the marked points, but depends on the choice of components and orientations. It is easy to see that the above product is well-defined, that is, for any links L_1, L_2, L_3 and L_4, if $L_1 \sim L_2$ and $L_3 \sim L_4$, then $L_1 \# L_3 \sim L_2 \# L_4$. Further it satisfies the associative law and the commutative law and the trivial knot acts as the identity element in $\mathcal{L}$. Thus $\mathcal{L}$ forms a commutative monoid.

In this paper, we represent links as words over the four letters (,), [and], and consider relations between them. This method is given by Manturov in [2] and we reformulate his result. For more information about links (or knots), we refer to [1].

2 Regular bibrackets and chord diagrams

Let Σ be an alphabet consisting of the four letters (,), [,]. A word A over Σ is called a *bibracket*. A bibracket A is called (,)-*regular* (resp. [,]-regular) if the number of opening and closing parentheses (resp. square brackets) are equal in A, and in every proper prefix of A, the number of opening parentheses (resp. square brackets) is larger than or equal to the number of closing parentheses (resp. square brackets). A bibracket is called *regular* if it is both (,)-regular and [,]-regular. A product of regular bibrackets is defined by a concatenation of words over Σ.

To construct a regular bibracket from a link, we need the following notion. A *chord diagram* is drawn on the plane as follows. First draw a circle and give a fixed point on the circumference, then draw straight lines (resp. curves) inside (resp. outside) the circle connecting two points not equal to the fixed point on the circumference in such a way that each pair of straight lines and curves do not intersect (see Fig.4). For a chord diagram C, straight lines inside (resp. curves outside) the circle are called *inner* (resp. *outer*) *chords* of C.

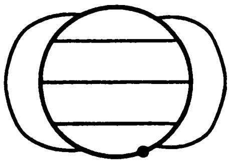


Fig.4

With each chord diagram C, we associate a regular bibracket A as follows. We trace the circle of C starting from the fixed point in the counterclockwise direction. If we encounter the first end point of an inner (resp. outer) chord, then we write an opening parenthesis (resp. square bracket) and, if we encounter the second end of it, we write a closing parenthesis (resp. square bracket). We proceed in this way and when we reach the start point, we spell out a bibracket A (see Fig.5) and it is easy to verify that it is actually a regular bibracket. Conversely, for a given regular bibracket A, we can easily construct a chord diagram C with which A associates.

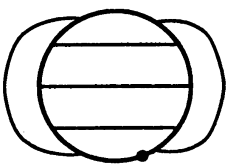
 $\longleftrightarrow$ [((( ] [ )))]

Fig.5

Next we construct a chord diagram from a given link L. We may suppose L is a link diagram on the plane. Split every crossing in L by the rule given in Fig.6.

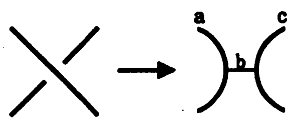


Fig.6

We assume that arcs a and c are parts of a circle of a chord diagram and b is an (inner or outer) chord. If we get only one circle when we split all the crossings, then we can get a chord diagram (see Fig.7).

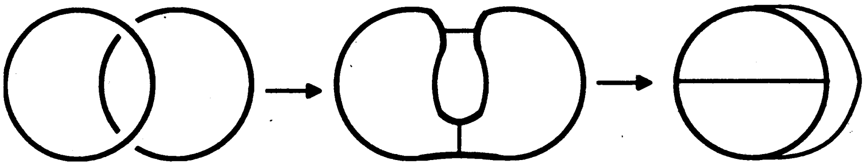


Fig.7

But the above case is exceptional, that is, when we split all the crossings, many circles may occur in general (see Fig.8).

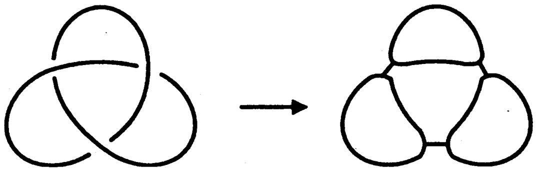


Fig.8

In this case, applying the deformation of Type 2, we can obtain a chord diagram as seen in Fig.9.

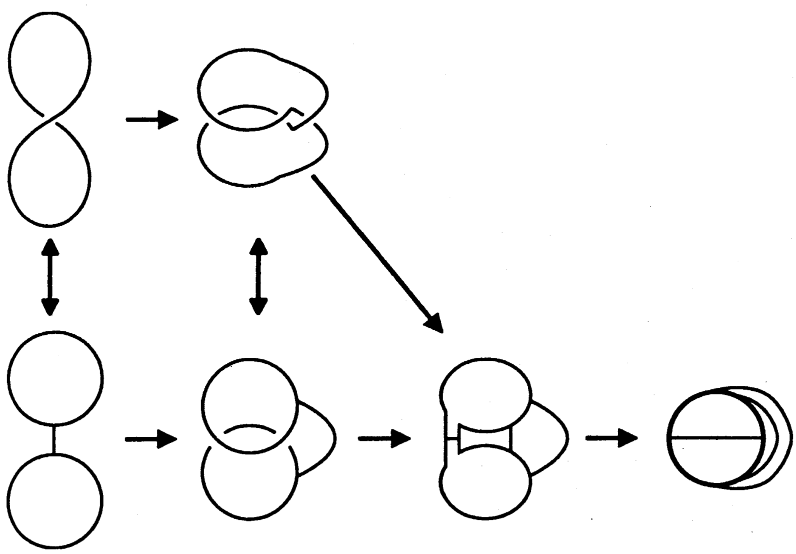


Fig.9

Thus, we obtain a chord diagram and a regular bibracket associated with it from a link diagram. Conversely, applying the above process in a reverse way, we can construct a link diagram from a given regular bibracket. Remark that the link diagram constructed from a regular bibracket is unique, but there are many different regular bibrackets, which are constructed from a link diagram. For each regular bibracket A, the link uniquely constructed from A is denoted by $L(A)$ and is called the link *represented*

3 Representation of the monoid of links

As we have seen in the previous section, every link (or link diagram) can be represented by some regular bibracket. In this section, we shall find a relationship between regular bibrackets which represent equivalent links. To this end, due to Result 1.1, it is sufficient to find relations between regular bibrackets which are transformed by the Reidemeister moves to each other. But since the Reidemeister moves are local deformations of links, we must consider the global situations where the Reidemeister moves apply.

The global Reidemeister moves can be obtained by closing end points of the Reidemeister moves with some parts of links as shown in Fig.10. These parts are denoted by A, B and C.

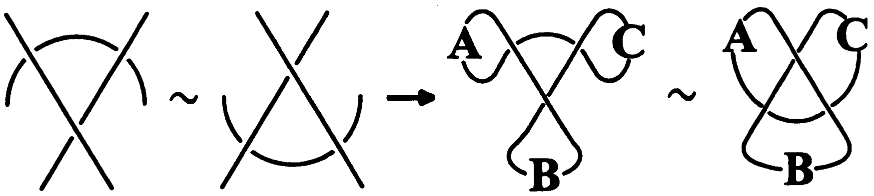


Fig.10

As described before, to obtain a regular bibracket from a link diagram, we may have to apply the deformation Type 2, but the way of the application is not unique. Accordingly we must consider the following three situations.

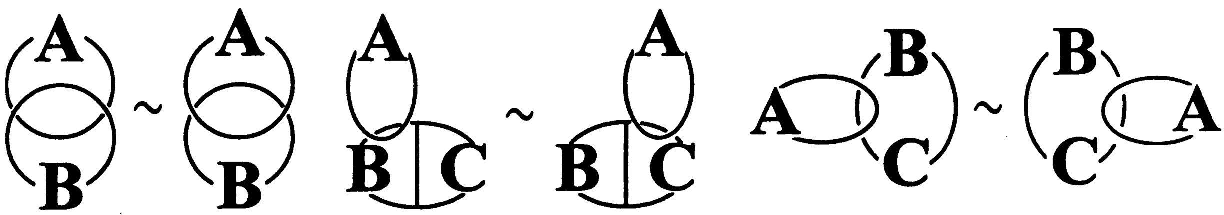


Fig.11

In Fig.11, A, B and C are parts of links, but without loss of generality, we may consider that A, B and C are bibrackets though they may not be regular. Then, from the leftmost equivalence of the links in Fig.11, we obtain the relation

$$A\,[\,(\,]\,B\,) = A\,(\,B\,[\,)\,]$$

through the process described in Fig.12.

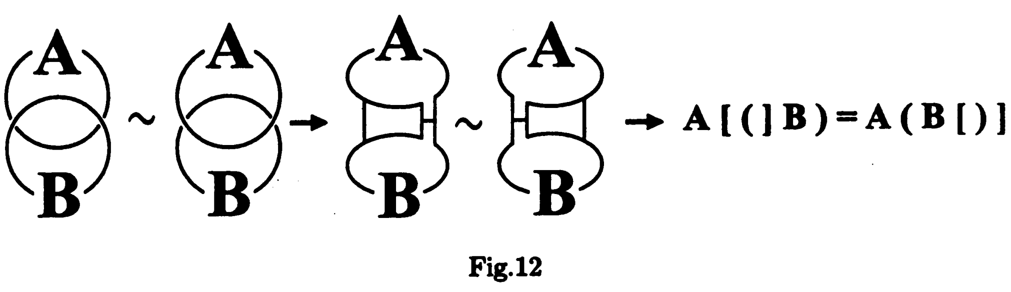


Fig.12

Similarly we obtain other relations between regular bibrackets from the other equivalences in Fig.11.

Next, we see the relations corresponding to the other types of the global Reidemeister moves. For example, in Fig.13, the equivalence at the top is the global Reidemeister moves of Type 3. At the second and third levels, the links are transformed to the chord diagrams. Finally, at the bottom, we have a relation between the regular bibrackets which associate with these links. In the same way, we can obtain all the relations from the other global Reidemeister moves, but details are omitted.

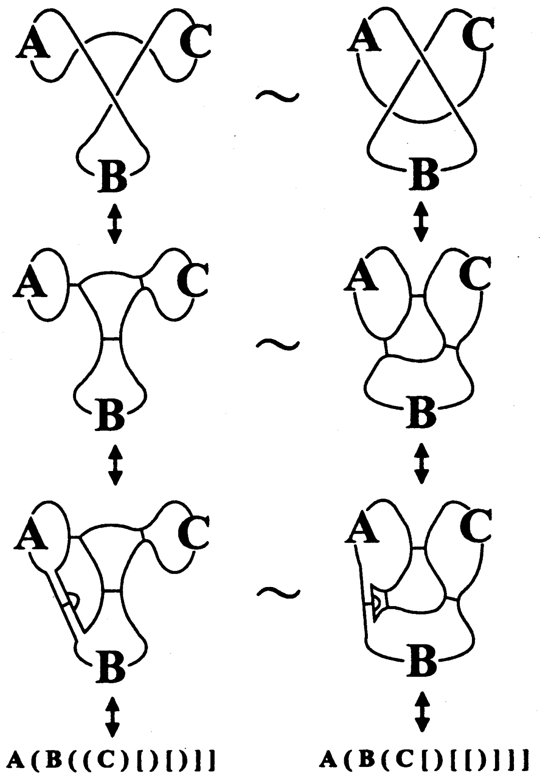


Fig.13

To describe the defining relations of the regular bibracket monoid, we need the following notions. For any regular bibracket A, $\overline{A}$ is a regular bibracket, which is obtained from A by changing all opening parentheses to opening square brackets, closing parentheses to closing square brackets and vice versa. It is easy to verify that links determined by A and $\overline{A}$ are equivalent. Thus we have a relation $A = \overline{A}$.

Next, for any regular bibracket of the form AB, where both A and B are bibrackets, the regular bibracket $B'A'$ is obtained in the following way. If BA is a regular bibracket, then $B'A'$ is equal to BA. If BA is not a regular bibracket, then there exists an opening parenthesis or an opening square bracket in A such that a corresponding closing parenthesis or square bracket is in B. If there is an opening parenthesis (resp. square bracket) in A such that the corresponding closing parenthesis (resp. square bracket) is in B, then we change this opening parenthesis (resp. square bracket) in A to the closing parenthesis (resp. square bracket) and the closing parenthesis (resp. square bracket) in B to the opening parenthesis (resp. square bracket). Applying the above replacement to all such pairs of parentheses and square brackets in BA, BA is transformed to a regular bibracket, which is denoted by $B'A'$. Since the product of links does not depend on the choice of a marked point, the links represented by AB and $B'A'$ are equivalent. Thus we include the relation $AB = B'A'$.

Now, we have our main theorem.

Theorem 3.1 *Let A and B be two regular bibrackets. Then two links $L(A)$ and $L(B)$ are equivalent if and only if one of A and B can be obtained from the other one by applying the following relations finitely many times.*

1. $A = \overline{A}$, *where A is regular,*

2. $AB = B'A'$, *where A, B are bibrackets and AB is regular,*

3. $A\ [\ (\ ]\ B\ ) = A\ (\ B\ [\ )\ ]$, *where A, B are $(,)$-regular and AB is regular,*

4. $A\ (\ (\ B\ )\ C\ [\ )\ ] = A\ (\ B\ (\ C\ )\ [\ )\ ]$, *where A, B, C are $(,)$-regular and ABC is regular,*

5. $A\ (\ B\ C\ [\ )\ ] = A\ (\ C'\ B'\ [\ )\ ]$, *where A, BC are $(,)$-regular, B, AC are $[,]$-regular and ABC is regular,*

6. $(\ ) = 1$ *where 1 is the empty word,*

7. $A\ (\ B\ [\ [\ )\ ]\ (\ ]\ ) = A\ B$, *where A, B are $(,)$-regular and AB is regular,*

8. $A\ (\ B\ (\ (\ C\ )\ [\ )\ [\ )\ ]\ ] = A\ (\ B\ (\ C\ [\ )\ [\ [\ )\ ]\ ]\ ]$, *where A, B, C are $(,)$-regular and ABC is regular,*

9. $A\ (\ [\ (\ B\ ]\ C\ )\ ) = A\ [\ (\ [\ B\ ]\ ]\ C\ )$, *where A, BC are $(,)$-regular, B, AC are $[,]$-regular and ABC is regular,*

10. $A\ (\ B\ (\ C\ (\ [\ [\ )\ [\ )\ ]\ ]\ [\ )\ ]\ ] = A\ [\ (\ (\ ]\ B\ )\ C\ )$, *where A, B, C are $(,)$-regular and ABC is regular,*

11. $A\ (\ [\ (\ B\ ]\ C\ [\ [\ )\ ]\ ] = A\ [\ (\ [\ B\ (\ (\ ]\ )\ )\ ]\ C\ )$, *where A, BC are $(,)$-regular, B, AC are $[,]$-regular and ABC is regular,*

12. $A\ (\ [\ (\ B\ ]\ C\ [\ [\ )\ ]\ ]\ ) = A\ [\ (\ [\ B\ ]\ (\ (\ ]\ )\ )\ C\ )$, *where A, BC are $(,)$-regular, B, AC are $[,]$-regular and ABC is regular.*

where 3, 4 and 5 come from the equivalences in Fig.11, 6 is from the Reidemeister moves of Type 1, 7 is from Type 2 and 8-12 are from Type 3.

References

[1] C. C. Adams, *The Knot Book - an elementary introduction to the mathematical theory of knots -*, W.H.FREEMAN AND COMPANY, New York, 1994.

[2] V. O. Manturov, *The Bracket Semigroup of Knots*, Mathematical Notes **67**, No.4 (2000) 468-478.